5. Agmon, S., Duglis, A. and Nirenberg, L., Calculation of Solutions Near the Boundary for Elliptic Partial Differential Equations with General Boundary Conditions. (Russian translation). M., Izd. inostr. lit., 1962.
6. Ditkin, V. A. and Prudnikov, A. P., Integral Transformations and Operational Calculus. M., Fizmatgiz, 1961.
7. Naimark, M. A., Linear Differential Operators. M., Gostekhizdat, 1954.
8. Ufliand, Ia. S., Integral Transformations for Problems in the Theory of Elasticity. M.-L., Izd. Akad. Nauk SSSR, 1963.

Translated by B. D.

ON THE METHOD OF ORTHOGONAL POLYNOMIALS IN CONTACT PROBLEMS OF THE THEORY OF ELASTICITY

PMM Vol. 33, №3, 1969, pp. 518-531
G. Ia. POPOV
(Odessa)
(Received October 3, 1968)

It is shown that the application of orthogonal polynomials to contact problems [1–12] is closely associated with the existence of a certain special calss of so-called polynomial kernels [13]. In [4, 6, 14, 15] particular cases of such kernels were constructed in various ways. Here we indicate a method of constructing polynomial kernels, on the basis of which not only all the previously constructed kernels may be obtained, but more general ones as well.

1. The essential features of the method of orthogonal polynomials. It is known that spatial contact problems with no friction force may be reduced to a two-dimensional integral equation of the first kind. To this equation must be adjoined a differential equation as well, if a plate rather than a stamp is being contacted. For contact regions such as a half-plane, strip, disk, or annulus, by means of some integral transformation or another, one may reduce the indicated two-dimensional system of equations to a one-dimensional problem. In the case of a stamp we thus have only a single one-dimensional integral equation of the first kind. In the case of a plate, however, we obtain a system composed of the equation indicated together with an ordinary differential equation. This last can likewise be reduced to an integral equation of the first kind by use of the Green's function for the differential equation obtained. One may get an idea of how this is done by looking at the example of a plane contact problem in [12].

Thus, spatial contact problems for the regions enumerated, and also plane problems with one contacting segment (sometimes two) may be reduced to solving an integral equation of the first kind

$$\int_a^b K(x, y)\varphi(y)\,dy = f(x) \qquad (a \leqslant x \leqslant b) \tag{1.1}$$

given on either a finite or a semi-infinite interval.

Such problems, but with account taken of the surface structure of the contacting bodies, were, in the formulation of Shtaerman [16], reduced to analogous integral equations of the second kind

$$\varphi(x) + \lambda \int_a^b K(x, y)\varphi(y)\,dy = f(x) \qquad (a \leqslant x \leqslant b) \tag{1.2}$$

In order to describe the essentials of the method of orthogonal polynomials as applicable to Eqs. (1.1) and (1.2), we introduce a special class of kernels according to the following definition. The function $\Pi(x, y)$ will be called a polynomial kernel, or in short a Π-kernel, on the interval (a, b) if the following relations hold:

$$\int_a^b \Pi_\pm(x, y)\, p_\pm(y)\, \pi_n^\pm(y)\,dy = \sigma_n g_\pm(x)\, \pi_n^\mp(x) \qquad (a \leqslant x \leqslant b,\ \sigma_n \neq 0;\quad n = 0, 1, 2, \ldots)$$

$$\Pi_+(x, y) = \Pi(x, y), \qquad \Pi_+(x, y) = \Pi(y, x) \tag{1.3}$$

Here $\pi_n^\pm(x)$ is a polynomial of degree exactly n, i.e. the coefficients of the highest powers $k_n^\pm \neq 0$, for definiteness we shall suppose that $\operatorname{Re} k_n^\pm > 0$. The polynomials indicated are orthogonal in the sense that

$$\int_a^b \pi_n^\pm(x)\, \pi_m^\pm(x)\, w_\pm(x)\,dx = \delta_{mn}, \qquad w_\pm(x) = p_\pm(x)\, g_\mp(x) \tag{1.4}$$

In this relation the weight functions may be complex. In the following relation (1.3) will for convenience be called the spectral relation (*).

If the weight function $w_\pm(x)$ is nonnegative and the condition

$$\int_a^b\int_a^b \left| \frac{p_-(x)}{g_+(x)}\, \Pi^2(x, y)\, \frac{p_+(y)}{g_-(y)} \right| dx\,dy < \infty \tag{1.5}$$

is fulfilled, then the Π-kernel will be said to be of Hilbert-type.

We introduce the kernel

$$H(x, y) = \left(\frac{p_-(x)}{g_+(x)}\right)^{1/2} \Pi(x, y) \left(\frac{p_+(y)}{g_-(y)}\right)^{1/2} \tag{1.6}$$

and the orthonormal system of functions

$$\varphi_n^\pm(x) = \sqrt{w_\pm x}\,\pi_n^\pm(x), \qquad \int_a^b \varphi_n^\pm(x)\, \varphi_m^\pm(x)\,dx = \delta_{mn} \tag{1.7}$$

Then (1.3) may be written in the form

$$\int_a^b H(x, y)\, \varphi_n^\pm(y)\,dy = \sigma_n \varphi_n^\mp(x), \qquad \int_a^b H(y, x)\, \varphi_n^\mp(y)\,dy = \sigma_n \varphi_n^\pm(x) \tag{1.8}$$

Now, it is seen that (1.5) is equivalent to the condition that $H(x, y)$ belong to the space L_2, so from (1.8) according to the theory of Schmidt (see e.g. [17]), the kernel $H(x, y)$ may be expanded in a bilinear series, involving the orthogonal functions $\varphi_n^\pm(x)$, and this series will converge in the mean. This means that the following bilinear expansion is always valid for Hilbert kernels:

$$\Pi(x, y) = g_+(x)\, g_-(y) \sum_{n=0}^{\infty} \sigma_n \pi_n^-(x)\, \pi_n^+(y) \tag{1.9}$$

Suppose now it is required to construct a solution of the integral equation of the first kind

---

*) This relation will actually determine the discrete spectrum of the integral operator with kernel $\Pi(x, y)$, multiplied by some weight function, if $\Pi(x, y) = \Pi(y, x)$.

$$\int_a^b \Pi(x, y)\,\varphi(y)\,dy = f(x) \qquad (a \leqslant x \leqslant b) \tag{1.10}$$

Expanding the right side in a series

$$f(x) = g_+(x) \sum_{n=0}^{\infty} f_n \pi_n^-(x), \qquad f_n = \int_a^b p_-(x)\,\pi_n^-(x)\,f(x)\,dx$$

and considering (1.3), we find formally

$$\varphi(x) = p_+(x) \sum_{n=0}^{\infty} \frac{f_n}{\sigma_n} \pi_n^+(x) \tag{1.11}$$

This will be also a solution in the strict sense (convergence understood in the sense of weighted mean convergence) of Eq. (1.10), if $\Pi(x, y)$ is a Hilbert kernel, and, moreover, the series constructed from the numbers $|f_n \sigma_n^{-1}|^2$, $n = 0, 1, 2\ldots$ converges. If the equation is given over a finite interval, then by virtue of the completeness in L_2 of the system of polynomials [18], the constructed solution will be also unique [17] in L_2. In the case of a semi-infinite interval this will be true in the case of weight functions which guarantee the completeness of the corresponding π-polynomials [18].

To solve the integral equation

$$\varphi(x) + \lambda \int_a^b \Pi(x, y)\,p(y)\,\varphi(y)\,dy = f(x) \tag{1.12}$$

with Hilbert Π-kernel one uses the bilinear expansion (1.9). As a result the solution is found in the form

$$\varphi(x) = f(x) + \lambda q_+(x) \sum_{m=0}^{\infty} \sigma_m \varphi_m \pi_m^-(x)$$
$$\left(\varphi_m = \int_a^b g_-(y)\,\pi_m^+(y)\,p(y)\,\varphi(y)\,dy\right) \tag{1.13}$$

In this the coefficients φ_m are found from the following infinite system of algebraic equations:

$$\varphi_n + \lambda \sum_{m=0}^{\infty} \sigma_m a_{nm} \varphi_m = f_n \qquad (n = 0, 1, 2, \ldots) \tag{1.14}$$

$$a_{nm} = \int_a^b g_+(x)\,g_-(x)\,p(x)\,\pi_n^+(x)\,\pi_m^-(x)\,dx, \qquad f_n = \int_a^b g_-(x)\,\pi_n^+(x)\,p(x)\,f(x)\,dx \tag{1.15}$$

obtained from (1.13) by multiplying it by $\pi_n^+(x) g_-(x) p(x)$ and integrating the result with respect to x from a to b. The solution constructed formally may be substantiated by drawing on Hilbert space methods.

We shall consider the interval to be finite and set

$$|\pi_n^+(x)| < A \qquad (n = 0, 1, 2, \ldots), \qquad \int_a^b \left| \frac{q_+(x)}{p_-(x)} p^2(x)\,g_-^2(x) \right| dx = J^2 < \infty \tag{1.16}$$

Then the infinite system will be completely regular, if

$$\lambda ABJ < 1 - \varepsilon \quad \left(B^2 = \sum_{n=0}^{\infty} |\sigma_n|^2\right) \tag{1.17}$$

The number $B < \infty$ by virtue of the Hilbert-like nature of the kernel [17]. This result follows from the estimate

$$\left(\sum_{m=0}^{\infty}|\sigma_m||a_{nm}|\right)^2 \leqslant B^2 \sum_{m=0}^{\infty}|a_{nm}|^2 = B^2 \int_a^b |h(x)\varphi_n^+(x)|^2 dx \tag{1.18}$$

The latter equation follows from the representation

$$a_{nm} = \int_a^b h(x)\varphi_n^+(x)\varphi_m^-(x), \qquad h^2(x) = \frac{g_+(x)g_-(x)}{p_+(x)p_-(x)}p(x)$$

obtained from (1.15) by going over to orthogonal functions (1.7). Precisely from this representation it is clear that a_{nm} are Fourier coefficients of the function $h(x)\varphi_n^+(x)$ which (by virtue of the second of conditions (1.16)) belongs to the space $L_2(a, b)$. Finally, an estimate of the integral in (1.18) with use of both conditions (1.16) leads to (1.17). The coefficients f_n $(n = 0,1,2)$ will be bounded in number, if

$$\sqrt{q_-(x)}\,[p_+(x)]^{-1/2} f(x)p(x) \in L_2(a, b),$$

since they will represent Fourier coefficients of the indicated function in the system $\varphi_n^+(x)$. Of course in specific cases one may find more refined regularity conditions [4]. We note that the method of orthogonal polynomials is also useful in seeking [19] the spectrum of Eq. (1.12).

One deals with integral equations of the type (1.10) in contact problems [4, 7, 9] for an elastic half-space (homogeneous or with modulus of elasticity variable according to a power law). These same problems, but with account taken of the surface structure of the contacting bodies, lead to equations of the type (1.12).

The application of the method of orthogonal polynomials to the general case (1.1) consists in the following. One splits off a Π-kernel from the kernel of Eq. (1.1), i.e.

$$K(x, y) = \Pi(x, y) + D(x, y) \tag{1.19}$$

and the solution is constructed in the form of a series

$$\varphi(x) = p_+(x)\sum_{m=0}^{\infty}\varphi_m \pi_m^+(x) \tag{1.20}$$

To seek the coefficients φ_m we must substitute (1.19) and (1.20) into (1.1). After use of the spectral relation (1.3), one multiplies both sides by $p_-(x)\pi_n^-(x)$ and integrates from a to b. As a result we obtain an infinite system of equations

$$\sigma_n\varphi_n + \sum_{m=0}^{\infty} d_{nm}\varphi_m = f_n \qquad (n = 0, 1, 2, \ldots) \tag{1.21}$$

$$d_{nm} = \int_a^b\int_a^b \frac{\pi_n^-(x)\pi_m^+(y)D(x, y)}{[p_-(x)p_+(y)]^{-1}}\,dxdy, \qquad f_n = \int_a^b p_-(x)\pi_n^-(x)f(x)\,dx$$

The justification of this formal procedure follows from Hilbert space methods which must be applied here. The condition of regularity of the system (1.21) in both cases is obtained with difficulty. However, for a sufficiently wide class of kernels, embracing a large class of contact problems, one may prove its quasi-complete regularity [11]. How to obtain a representation (1.19) in practice is shown in [2–5, 7, 8, 11]. In applications to contact problems it proves always to be the case that the Π-kernel carries the singularity of the kernel of the equation, and the function $D(x, y)$ in the worst case turns out to be continuous, and even differentiable any number of times.

It may easily be seen that the coefficients d_{nm} determine a representation of the function $D(x, y)$ in the form of a double series in the $\pi_n^{\pm}$-polynomials. If one retains a finite number of terms in this series, one comes up with the following approximation:

$$D(x, y) \approx g_+(x) g_-(y) \sum_{m=0}^{N} \sum_{n=0}^{N} d_{nm} \pi_n^-(x) \pi_m^+(y) \tag{1.22}$$

Then, clearly, to find approximate values for φ_m $(m = 0, 1, .., N)$, we have the following system of equations:

$$\sigma_n \varphi_n + \sum_{m=0}^{N} d_{nm} \varphi_m = f_n \qquad (n = 0, 1, 2, \ldots, N) \tag{1.23}$$

and for the remaining φ_m $(m = N+1, N+2, \ldots)$, the formula $\varphi_m = f_m \sigma_m^{-1}$. Now, if the function $f(x) / g_+(x)$ is a polynomial of degree M, which is fairly often the case in contact problems [2, 3], the series (1.20) terminates at the term with index equal to max (N, M).

The approximation (1.22) does not always prove to be convenient. In these cases it may prove useful to approximate the function $D / g_+ g_-$ by interpolation polynomials, as is done in [12]. Moreover, in many contact problems [3, 4] an approximation of the following sort occurs naturally:

$$D(x, y) \approx g_+(x) g_-(y) \sum_{k=0}^{N} A_k P_k(x, y), \qquad P_k(x, y) = \sum_{j=0}^{k} a_{kj} x^{k-j} y^j$$

In this case system (1.23) transforms into

$$\sigma_n \varphi_n + \sum_{m=0}^{N-n} b_{nm} \varphi_m = f_n \qquad (n = 0, 1, 2, \ldots, N) \tag{1.24}$$

Thus such an approximation has the advantage over the first two, that the matrix of coefficients of the system of algebraic equations is almost triangular.

We note papers [6, 10], where the method of orthogonal polynomials is applied to the solution of (1.1) given on a finite interval, and at the same time the kernel of the equation is a Π-kernel on a semi-infinite interval.

From the above it is clear how important the question of the construction of the Π-kernel is. The following is concerned entirely with this question.

2. One auxiliary theorem on Π-kernels. We concern ourselves here with the proof of the following assertion.

Theorem 2.1. For the function $\Pi(x, y)$ to be a Π-kernel (on the interval (a, b)), it is necessary and sufficient for the following conditions to be satisfied:

a) there exist functions $p_\pm(x)$, $g_\pm(x)$ such that

$$\int^{b} \Pi_\pm(x, y) p_\pm(y) y^m dy = g_\pm(x) \sum_{j=0}^{m} b_{jm}^{\pm} x^j, \qquad b_{mm}^{\pm} \neq 0 \qquad (m = 0, 1, 2, \ldots) \tag{2.1}$$

b) the following equation holds:

$$\int_a^b x^n p_-(x)\, dx \int_a^b \Pi(x, y) p_+(y) y^m dy = \int_a^b p_+(y) y^m dy \int_a^b \Pi(x, y) p_-(x) x^n dx \tag{2.2}$$

c) there exist moments

$$c_n^\pm = \int_a^b w_\pm(x) x^n dx \qquad (n = 0, 1, 2, \ldots) \qquad (w_\pm(x) = p_\pm(x) g_\mp(x)) \tag{2.3}$$

d) the following determinants do not vanish:

$$D_n^{\pm}=\begin{vmatrix} c_0^{\pm} & c_1^{\pm},\ldots, c_n^{\pm} \\ c_1^{\pm} & c_2^{\pm},\ldots, c_{n+1}^{\pm} \\ c_n^{\pm} & c_{n+1}^{\pm},\ldots, c_{2n}^{\pm}\end{vmatrix} \neq 0 \qquad (n=0,1,2,\ldots) \tag{2.4}$$

We prove first the necessity of these conditions. Since $\pi^{\pm}_n(x)$ is of degree n exactly, then ([18], p. 17)

$$x^n=\sum_{j=0}^{n}\lambda_{nj}^{+}\pi_j^{\pm}(x), \qquad \lambda_{nn}^{\pm}\neq 0 \tag{2.5}$$

Consequently, on the basis of (1. 3) we may write

$$\int_a^b \Pi_{\pm}(x, y)\,p_{\pm}(y)\,y^n dy=g_{\pm}(x)\sum_{j=0}^{n}\lambda_{nj}^{\pm}\sigma_j\pi_j^{\mp}(x) \tag{2.6}$$

From this follows the necessity of condition (a), in which

$$b_{nn}^{\pm}=\lambda_{nn}^{\pm}k_n^{\mp}\sigma_n\neq 0 \qquad (n=0,1,2,\ldots) \tag{2.7}$$

In order to see the necessity of (b), one must write the iterated integrals in (2. 2) by use of (2. 6) and (2. 5). The necessity of the remaining conditions arises from the following lemma.

Lemma. If for some linear functional and some system of polynomials $\pi_n(x)= = k_n x^n+\ldots, k_n\neq 0$ $(n=0, 1, 2,\ldots)$ we have the relation

$$F\{\pi_j(x)\pi_k(x)\}=\delta_{jk} \tag{2.8}$$

then there exist numbers (moments) $F\{x^n x^\circ\}=c_n$, for which

$$D_n=\det\|c_{j+k}\|_0^n\neq 0 \qquad (n=0, 1, 2,\ldots)$$

In fact, by virtue of (2. 8), (2. 5) and the linearity of the functional, we have

$$F\{x^n x^\circ\}=F\left\{\frac{x^n\pi_0(x)}{k_0}\right\}=F\left\{\frac{\pi_0(x)}{k_0}\sum_{j=0}^{n}\lambda_{nj}\pi_j(x)\right\}=\frac{\lambda_{n0}}{k_0}=c_n$$

On the other hand

$$c_{j+k}=F\{x^j x^k\}=F\left\{\sum_{r=0}^{j}\lambda_{jr}\pi_r(x)\sum_{s=0}^{k}\lambda_{ks}\pi_s(x)\right\}=\sum_{r=0}^{\min(k,j)}\lambda_{jr}\lambda_{kr} \tag{2.9}$$

If we now introduce the triangular matrix

$$\Lambda_n=\begin{Vmatrix}\lambda_{00} & 0 & 0 & \ldots & 0\\ \lambda_{10} & \lambda_{11} & 0 & \ldots & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \lambda_{n0} & \lambda_{n1} & \lambda_{n2} & \ldots & \lambda_{nn}\end{Vmatrix}, \qquad \det\Lambda_n=\prod_{j=0}^{n}d_{jj}\neq 0$$

and its transpose Λ_n', $\det\Lambda_n'=\det\Lambda_n$, then the equation (2. 9) obtained means nothing other than $\|c_{j+k}\|_0^n=\Lambda_n, \Lambda_n'$ and therefore

$$\det\|c_{j+k}\|_0^n=\det\Lambda_n,\ \det\Lambda_n'\neq 0$$

We pass to the proof of the sufficiency of the conditions of Theorem 2. 1. For this we multiply both sides of the first of relations (2. 1) by $p_-(x)x^n$ $(n=0, 1, 2,\ldots)$ and integrate with respect to x from a to b, after which on the left side we represent the integral on the basis of (2. 2). As a result of using the second relation from (2. 1) and the notation (2. 3) we obtain

$$\sum_{j=0}^{n}b_{jn}^{-}c_{m+j}^{+}=\sum_{j=0}^{m}b_{jm}^{+}c_{n+j} \tag{2.10}$$

Furthermore, using condition (d), we construct ([18], p. 416) two families of polynomials

$$\pi_n^{\pm}(x) = \frac{1}{\sqrt{D_n^{\pm} D_{n-1}^{\pm}}} \begin{vmatrix} c_0^{\pm} & c_1^{\pm} & \ldots & c_n^{\pm} \\ c_1^{\pm} & c_1^{\pm} & \ldots & c_{n+1}^{\pm} \\ \ldots & \ldots & \ldots & \ldots \\ c_{n-1}^{\pm} & c_n^{\pm} & \ldots & c_{2n-1}^{\pm} \\ 1 & x & \ldots & x^n \end{vmatrix}, \qquad k_n^{\pm} = \left(\frac{D_{n-1}^{\pm}}{D_n^{\pm}}\right)^{1/2} \tag{2.11}$$

satisfying condition (1. 4).

Denoting the cofactors of the terms in the last row of the determinant in (2. 11) by $A_{jn}^{\pm}$ $(j = 0,1,2,\ldots,n)$, we may write

$$\pi_n^{\pm}(x) = \frac{1}{\sqrt{D_n^{\pm} D_{n-1}^{\pm}}} \sum_{j=0}^{n} A_{jn}^{\pm} x^j \tag{2.12}$$

Using the last formula, and also the first relation of (2. 1), we calculate the integral

$$\int_a^b \Pi(x, y)\, p_+(y)\, \pi_n^+(y)\, dy = g_+(x)\, P_n^-(x)$$

$$P_n^-(x) = \sum_{j=0}^{n} \frac{A_{jn}^+}{\sqrt{D_n^+ D_{n-1}^+}} \sum_{r=0}^{j} b_{rj}^+ x^r \tag{2.13}$$

It now remains to be shown that the polynomial (2. 13) differs by only a constant from $\pi_n^-(x)$ determined by formula (2. 11) or (2. 12), i. e. $P_n^-(x) = \sigma_n^- \pi_n^-(x)$, $\sigma_n^- \neq 0$. As is known [18], for this it suffices to show that

$$J_n = \int_a^b P_n^-(x)\, x^m w_-(x)\, dx = 0 \qquad (m = 0,\ 1,\ 2, \ldots,\ n-1)$$

Substituting expression (2. 13) into the left of the latter equation and taking (2. 10) into consideration, we obtain

$$J_n = \sum_{j=0}^{n} A_{jn}^+ \sum_{r=0}^{j} b_{rj}^+ \frac{c_{m+r}}{\sqrt{D_n^+ D_{n-1}^+}} = \sum_{r=0}^{m} \frac{b_{rm}^-}{\sqrt{D_n^+ D_{n-1}^+}} \sum_{j=0}^{n} A_{jn}^+ c_{j+r}^+ = 0 \qquad (m < n)$$

The last equation is valid since the A_{jn}^+ are the cofactors of the entries in the last row of the determinant in (2. 11). Finally, equating the coefficients of x^n in the polynomials $P_n^-(x)$ and $\pi_n^-(x)$, we find $\sigma_n^- = b_{nn}^+ k_n^+ [k_n^-]^{-1} \neq 0$, since b_{nn}^+, $k_n^+ \neq 0$. Thus it is proved that

$$\int_a^b \Pi(x, y)\, p_+(y)\, \pi_n^+(y)\, dy = \sigma_n^- g_+(x)\, \pi_n^-(x) \tag{2.14}$$

In an analogous manner we may clearly prove the following:

$$\int_a^b \Pi(x, y)\, p_-(x)\, \pi_n^-(x)\, dx = \sigma_n^+ g_-(y)\, \pi_n^+(y) \tag{2.15}$$

where $\sigma_n^+ = b_{nn}^- k_n^- [k_n^+]^{-1} \neq 0$.

It now remains to be shown that

$$b_{nn}^- k_n^- \,[k_n^+]^{-1} = b_{nn}^+ k_n^+ \,[k_n^-]^{-1} = \sigma_n \tag{2.16}$$

But this equation is a consequence of condition (b). To see this, one must multiply both sides of (2. 14) by $\pi_n^-(x) p_-(x)$, integrate with respect to x from a to b, change the order of integration in the left sides of the basis of (2. 2) and, finally, compare with the equation obtained from (2.15) by integrating the latter with respect to x with the weight $\pi_n^+(y)p_+(y)$ in the interval (a, b).

The theorem is thereby completely proved.

We make a few remarks. The theorem is valid also when in place of (a, b) we take an arbitrary line (or several lines) in the complex plane. For nonnegative weight functions $w_\pm(x)$ condition (d) of the theorem is fulfilled automatically ([18], p. 39); and, finally, in the case of a Hilbert kernel, conditions (2.1) and (1.3) are equivalent. One may see this introducing the kernel (1.6) and the associated iterated Schmidt kernel, using the assertions of Hilbert-Schmidt theory [17] and the considerations given at the beginning of [4].

3. Construction of a Π-kernel on a finite interval. We introduce into consideration a kernel

$$K(x, y) = \int_a^{\min(x,y)} K_+(x-s)K_-(y-s)\rho(s)\,ds \quad (a \leqslant x, y \leqslant b) \tag{3.1}$$

and concern ourselves with the question of what the functions K_+, K_-, ρ should be in order for it to be a Π-kernel on the interval (a, b). For this we consider the integral transforms

$$J_+ = \int_a^b K(x, y)\varphi_+(y)\,dy, \quad J_- = \int_a^b K(y, x)\varphi_-(y)\,dy \tag{3.2}$$

where $\varphi_\pm(x)$ are for the time being arbitrary functions.

We divide the interval of integration in formulas (3.2) into two parts: (a, x) and (x, b), using (3.1) therein. As a result of interchanging the integrals, which will be justified later, we shall have

$$J_\pm = \int_a^x \rho(s)K_\pm(x-s)\,ds \int_s^b K_\mp(y-s)\varphi_\pm(y)\,dy$$

After obvious changes of variables we obtain

$$\int_0^1 \xi\rho(\xi\tau+a)K_\pm[\xi(1-\tau)](b-a-\xi\tau)\,d\tau \int_0^1 K_\mp[t(b-a-\xi\tau)]\varphi_\pm \times$$

$$\times[\xi\tau + a + t(b-a-\xi\tau)]\,dt = J_\pm \qquad (\xi = x-a) \tag{3.3}$$

We now put

$$\varphi_\pm(y) = y^n p_\pm(y) \qquad (n = 0, 1, 2, \ldots) \tag{3.4}$$

Then it will be easy to see from (3.3), that the relation (2.1) will hold, if

$$\xi\rho(\xi\tau+a)K_\pm[\xi(1-\tau)](b-a-\xi\tau)K_\mp[t(b-a-\xi\tau)]p_\pm[\xi\tau+a+t(b-a-$$

$$-\xi\tau)] = g_\pm(x)F_\pm(\tau, t) \qquad (\xi = x-a) \tag{3.5}$$

It is easy to see that the latter will be fulfilled, when

$$\rho(s) = (s-a)^\rho(b-s)^\sigma, \quad K_+(u) = u^{-\alpha} \quad K_-(u) = u^{-\beta}, \quad p_\pm(y) = (b-y)^{-\beta\pm} \tag{3.6}$$

Here

$$\beta_+ = 1+\sigma-\beta, \qquad \beta_- = 1+\sigma-\alpha \tag{3.7}$$

$$g_+(x) = (x-a)^{1+\rho-\alpha}, \qquad g_-(x) = (x-a)^{1+\rho-\beta} \tag{3.8}$$

$$\begin{gathered} F_+(\tau, t) = \tau^\rho(1-\tau)^{-\alpha}t^{-\beta}(1-t)^{\beta-\sigma-1} \\ F_-(\tau, t) = \tau^\rho(1-\tau)^{-\beta}t^{-\alpha}(1-t)^{\alpha-\sigma-1} \end{gathered} \tag{3.9}$$

As a result of calculating the integrals in (3.3) with account taken of (3.4)–(3.9), we find

$$b_{nn}^{+}=\frac{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\beta-\sigma+n)\,\Gamma(1+\rho+n)}{\Gamma(2+\rho-\alpha+n)\,\Gamma(1-\sigma+n)} \tag{3.10}$$

The formula for b_{nn}^{-} is obtained from (3.10) by interchanging the positions of the parameters α and β. Thus, on the basis of the above, the function

$$\Pi^*(x,y)=\int_a^c \frac{(s-a)^\rho (b-s)^\sigma}{(x-s)^\alpha\,(y-s)^\beta}\,ds \qquad \begin{matrix}(\operatorname{Re}(1+\rho,\ 1+\sigma,\ 1-\alpha,\ 1-\beta)>0)\\ c=\min(x,y)\end{matrix} \tag{3.11}$$

may prove to be a Π-kernel. In order to convince oneself of that, one should verify the applicability of all conditions of Theorem 2.1. The condition $b_{nn}^{\pm}\neq 0$ $(n=0,1,2)$ and condition (c) of the theorem will be fulfilled if, besides the limitations imposed on the parameters in (3.11), one requires in addition $\operatorname{Re}(\beta-\sigma,\ \alpha-\sigma)>0$ $(\sigma\neq 1,2,3)$.

Hence we arrive at the following condition:

$$\operatorname{Re}(1+\rho,\ 1+\sigma,\ 1-\alpha,\ 1-\beta,\ \beta-\sigma,\ \alpha-\sigma)>0 \qquad (\sigma\neq 1,2,3,\ldots) \tag{3.12}$$

It is easy to verify that these conditions guarantee that (2.2) is satisfied, and at the same time make valid the interchange of integrations performed in obtaining (3.3) and (3.5). It remains to be shown that condition (2) of Theorem 2.1 holds. Together with this we show the existence of π-polynomials for the kernel (3.11) considered. For this, it suffices,on the basis of (3.6)–(3.8), to indicate polynomials satisfying the following conditions of orthonormality:

$$\int_a^b \frac{\pi_n^{+}(x)\,\pi_m^{+}(x)\,dx}{(b-x)^{1+\sigma-\beta}\,(x-a)^{\beta-\rho-1}}=\int_a^b \frac{\pi_n^{-}(x)\,\pi_m^{+}(x)\,dx}{(b-x)^{1+\sigma-\alpha}\,(x-a)^{\alpha-\rho-1}}=\delta_{mn} \tag{3.13}$$

Considering the properties of the Jacobi polynomials $P_n^{\lambda,\mu}(z)$ (see e.g. [18, 20]), it may be seen that the first orthonormality condition (3.13) will be satisfied by the polynomial

$$\pi_n^{+}(x)=\delta_n^{+}P_n^{\beta-\sigma-1,\,1+\rho-\beta}\left(\frac{2x-a-b}{b-a}\right)$$
$$(\delta_n^{+})^2=\frac{n!\,\Gamma(1+\rho-\sigma+n)(1+\rho-\sigma+2n)}{(b-a)^{1+\rho-\sigma}\Gamma(2+\rho-\beta+n)\,\Gamma(\beta-\sigma+n)} \tag{3.14}$$

However the formula for $\pi_n^{-}(x)$ is obtained from (3.14) by interchanging the parameters α and β. Thus, the function (3.11) is actually a Π-kernel under conditions(3.12). To calculate the σ-numbers of this kernel one should use formula (2.16) with account taken of (3.10) and (3.14). Substituting into the first spectral relation (1.3) the elements for a Π-kernel (3.11), determined by formulas (3.6)–(3.8), we shall have

$$\int_a^b \frac{\Pi^*(x,y)}{(b-y)^{1+\sigma-\beta}}\,P_n^{\beta-\sigma-1,\,1+\rho-\beta}\left(\frac{2y-a-b}{b-a}\right)dy=$$
$$=\frac{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\beta-\sigma+n)\,\Gamma(1+\rho+n)\,P_n^{\alpha-\sigma-1,\,1+\rho-\alpha}}{\Gamma(1-\sigma+n)\,\Gamma(2+\rho-\alpha+n)\,(x-a)^{\alpha-\rho-1}}\left(\frac{2x-a-b}{b-a}\right) \tag{3.15}$$

The second spectral relation is obtained from (3.15) by interchanging the parameters α and β and replacing $\Pi^*(x,y)$ by $\Pi^*(y,x)$.

We note that the Π-kernel (3.11) may be expressed through the first function [20, 21] of Appell $F_1(\sigma,\beta,\beta',\gamma;\ x,y)$. Actually, considering individually in (3.11) the cases: $x<y$, $y<x$ and taking into consideration the known [21] integral representation for

the function mentioned, we obtain (3.16)

$$\Pi^*(x, y)=\frac{\Gamma(1+\rho)\Gamma(1-\alpha)(x-a)^{1-\alpha+\rho}}{\Gamma(2+\rho-\alpha)(y-\alpha)^{\beta}(b-a)^{-\sigma}}F_1\left(1+\rho, -\sigma, \beta, 2+\rho-\alpha; \frac{x-a}{b-a}, \frac{x-a}{y-a}\right)$$

In the case $y<x$ in (3.16) one should replace α by β and x by y.

It is of interest to clarify under which restrictions on the parameters the Π-kernel (3.11) obtained will in fact be a Hilbert kernel. From the definition it follows that for this it is necessary to have nonnegative weight functions $w_{\pm}(x)$ and the fulfilment of condition (1.5). In the case considered, by virtue of (3.6)–(3.8), the nonnegativity will be guaranteed if

$$\operatorname{Im}(\alpha-\sigma, \beta-\sigma, \alpha-\rho, \beta-\rho)=0 \tag{3.17}$$

However, the condition (1.5) for the kernel $\Pi^*(x, y)$ may be written in the following form, if one bears in mind the associated function $H^*(x, y)$ from formula (1.6):

$$\int_a^b\int_a^b |H^*(x, y)|^2 dx dy<\infty \tag{3.18}$$

For $H^*(x, y)$ the following representation holds:

$$H^*(x, y)=\int_a^b H_\alpha(x, t) H_\beta(y, t)\, dt \tag{3.19}$$

$$H_\gamma(x, y)=\left(\frac{(b-x)^{\gamma-\sigma-1}(b-y)^{\sigma}}{(x-a)^{1+\rho-\gamma}(y-a)^{-\rho}}\right)^{1/2}\begin{cases}(x-y)^{-\gamma}, & x>y\\ 0, & x<y\end{cases}\qquad (\gamma=\alpha, \beta)$$

which is seen by direct substitution to be valid. By virtue of the Cauchy-Buniakowski inequality applied to the integral in (3.14), for the validity of (3.18) it is sufficient that

$$\int_a^b\int_a^b |H_\gamma(x, y)|^2 dx dy<\infty \qquad (\gamma=\alpha, \beta)$$

This integral may be evaluated when the conditions (3.12) and $2\operatorname{Re}\gamma<1$ are fulfilled, if after substituting into it the expression for $H_\gamma(x, y)$, one performs a substitution changing the interval of integration from (a, b) to $(0, 1)$, and uses formulas 9.111 and 7.512 (b) from [20].

Hence, when (3.12) and (3.17) are satisfied, and also $2\operatorname{Re}(\alpha, \beta)<1$, the function (3.11) or (3.16) will be a Hilbert Π-kernel and therefore

$$\Pi^*(x, y)=\Gamma(1-\alpha)\Gamma(1-\beta)(b-a)^{\sigma-\rho-1}(x-a)^{1-\alpha+\rho}(y-a)^{1-\beta+\rho}\times$$

$$\times\sum_{n=0}^{\infty}\frac{n!\,\Gamma(1-\sigma+\rho+n)\Gamma(1+\rho+n)P_n^{\alpha-\sigma-1,1+\rho-\alpha}(u)P_n^{\beta-\sigma-1,1+\rho-\beta}(v)}{\Gamma(1-\sigma+n)\Gamma(2-\alpha+\rho+n)\Gamma(2-\beta+\rho+n)(1-\sigma+\rho+2n)^{-1}} \tag{3.20}$$

$$(a\leqslant x, y\leqslant b)$$

Here and throughout the following we use

$$u=(b-a)^{-1}(2x-a-b), \qquad v=(b-a)^{-1}(2y-a-b)$$

Thanks to the presence in the construction of the Π-kernel of a large number of parameters, we may obtain from it a wide choice of Π-kernels, including all the previously found Π-kernels on a finite interval. We begin with the case $\sigma=\alpha+\beta-\rho-2$, when the Appell function in (3.15) reduces ([21], p. 231) to the Gauss function $F(\alpha, \beta; \gamma; x)$. Subsequent introduction of parameters

$$\mu=1+\rho-\alpha, \qquad \gamma=1+\rho-\beta, \qquad \nu=\alpha+\beta-1 \tag{3.21}$$

and use of formula 6.574 from [20] (assuming $\operatorname{Re}(1-\nu)>0$) leads to the following Π-kernel:

$$\frac{\Pi(x,y)}{\Gamma(1-\alpha)\Gamma(1-\beta)}=\left(\frac{b-a}{2}\right)^{\nu}\left(\frac{y-a}{b-x}\right)^{1/2\gamma}\left(\frac{x-a}{b-y}\right)^{1/2\mu}\times$$
$$\times W^{\nu}_{\mu,\gamma}\left(\sqrt{(x-a)(b-y)},\quad \sqrt{(b-x)(y-a)}\right) \tag{3.22}$$

Here

$$W^{\nu}_{\mu,\gamma}(\xi,\eta)=\int_0^{\infty} t^{\nu} J_{\mu}(t\xi)\, J_{\gamma}(t\eta)\,dt$$

is the so-called discontinuous integral of Weber-Schafheitlin [22]. For Π-kernel (3.22) the spectral relation and bilinear expansion are obtained from (3.15) and (3.20), if in place of (3.16) we take (3.22) and bear in mind that $\sigma=\alpha+\beta-\rho-2$ and (3.21) holds. In this the limitations of the parameters will have the form

$$\operatorname{Re}[1+\gamma,\ 1+\mu,\ 1+\nu+\mu+\gamma,\ 1+\nu-\mu-\gamma,\ 1-\nu+\gamma-\mu,\ 1-\nu+\mu-$$
$$-\gamma,\ 1-\nu]>0$$

$$\operatorname{Re}(1+\nu-\mu+\gamma,\ 1+\nu-\gamma+\mu)<1,\qquad \operatorname{Im}(\mu,\gamma)=0 \tag{3.23}$$

It is necessary to consider the last two restrictions only in the case of a Hilbert kernel (3.22). For the case $a=0$, $b=1$, the spectral relation and bilinear expansion for the Π-kernel (3.22) are formally obtained in [14]. One may also find there many particular cases. If in (3.16) we now set $\sigma=0$ and take (3.21), then we come to the following Π-kernel:

$$\frac{\Pi(x,y)}{\Gamma(1-\alpha)\Gamma(1-\beta)}=\frac{1}{2^{\nu}}\frac{(x-a)^{1/2\mu}}{(y-a)^{-1/2\gamma}}W^{\nu}_{\mu,\gamma}\left(\sqrt{x-a},\ \sqrt{y-a}\right)\quad (\operatorname{Re}(1-\nu)>0) \tag{3.24}$$

The spectral relation and bilinear expansion for this Hilbert kernel (if the conditions $2\operatorname{Re}(\alpha,\beta)<1$, (3.12) and (3.17) are fulfilled) follow from (3.15) and (3.20) for $\sigma=0$ and (3.21). They were shown for the case $a=0$, $b=1$ in [14]. There, and also in [4], one may find numerous particular cases of Π-kernels (3.24), many of which found application in contact problems [4]. However the bilinear expansions obtained in the papers cited were not justified there, because of the absence of proof of the Hilbert-like nature of the corresponding kernels. The condition of Hilbert-likeness obtained here, $2\operatorname{Re}(\alpha,\beta)<1$, (3.12 and (3.17), and their particular case (3.23) fill in this gap (to a certain extent).

Without dwelling on the various particular cases of Π-kernels (3.24) in the interest of saving space, we indicate only one important case, which although contained in [4], was not remarked upon. We arrive at this case if in (3.24) we set $a=0$, $b=1$, $\mu=\gamma=1/2$, $\nu=0$, $x=\xi^2$, $y=\eta^2$, and consider formulas 6.576 (2) and 9.121 (6) of [20]. The spectral relation and bilinear expansion corresponding to it

$$\int_0^1 \frac{T_{2n+1}(\eta)}{\sqrt{1-\eta^2}}\ln\frac{\xi+\eta}{|\xi-\eta|}\,d\eta=\frac{\pi}{1+2n}T_{2n+1}(\xi)$$

$$\ln\frac{\xi+\eta}{|\xi-\eta|}=\sum_{n=0}^{\infty}\frac{4}{1+2n}T_{2n+1}(\xi)\,T_{2n+1}(\eta)$$

$(0\leqslant\xi,\ \eta\leqslant 1,\ T_n(x)$ are Chebyshev polynomials)

we obtain from (3.15) and (3.20), fixing the parameters as indicated. This bilinear expansion was used in the solution of a problem in hydrodynamics in [19].

In order to obtain the Π-kernels constructed in [15], we must set $\sigma = 0$, $\beta = 2 + \rho - \alpha$, $1 + \rho = \nu$ in (3.16) or (3.11). As a result we find the Π-kernel

$$\Pi(x, y) = \frac{\Gamma(\nu)\Gamma(\alpha - \nu)(y - a)^{\alpha-1}}{\Gamma(\alpha)|x - y|^{\nu}(x - a)^{\alpha-\nu}} \begin{cases} \sin\pi(\alpha - \nu)\operatorname{cosec}\pi\alpha, & x < y \\ 1, & y < x \end{cases}$$

which may also be written in the form

$$\Pi(x, y) = \frac{\Gamma(\nu)\Gamma(\alpha - \nu)[a_1 \operatorname{sign}(x - y) + a_2]^{\mu}(y - a)^{\alpha-1}}{\Gamma(\alpha)(a_2 + a_1)^{\mu}(x - a)^{\alpha-\nu}|x - y|^{\nu}} \tag{3.25}$$

Here

$$\alpha = \frac{1}{\pi}\arcsin\left(\frac{\sin\pi\nu}{A}\right), \quad A = \sqrt{1 - 2a_*^{\mu}\cos\pi\nu + a_*^{2\mu}} \quad \left(a_* = \frac{a_2 - a_1}{a_2 + a_1}\right)$$

The parameters a_1, a_2, μ in this expression, and also the branch of the arcsine function should be chosen so that the condition $0 < \operatorname{Re}\alpha < 1$ is guaranteed. Namely this Π-kernel for $a = 0$, $b = 1$ was in fact constructed in [15], where one may also find other Π-kernels, obtained from (3.25) by passing to particular values of the parameters. The spectral relation and bilinear expansion indicated in the cited work follow, of course, from (3.15) and 3.20). An application of the Π-kernel (3.25) to a contact problem was given in [9].

Setting $\sigma = 0$, $\rho = \alpha + \beta - 2$($\rho = \nu - 1$, $\beta = 1 + \nu - \alpha$, $\nu = \alpha + \beta - 1$) in (3.15) then, taking (3.16) into consideration, we shall have

$$\int_a^b \frac{[a^+ + a^- \operatorname{sign}(x - y)] P_n^{\nu-\alpha,\,\alpha-1}(v)}{|x - y|^{\nu}(y - a)^{1-\alpha}(b - y)^{\alpha-\nu}}\,dy = \frac{\pi(\nu)_n P_n^{\alpha-1,\,\nu-\alpha}(u)}{n!\sin\pi\alpha\sin\pi(\alpha - \nu)}$$

$$(a^{\pm} = [2\sin\pi\alpha\sin\pi(\alpha - \nu)]^{-1}\ [\sin\pi\alpha \pm \sin\pi(\alpha - \nu)])$$

If we use the orthogonality of the Jacobi polynomials, the relation obtained may be written in the form

$$\int_a^b \left[\frac{1}{|x - y|^{\nu}} - 1\right]\frac{1}{\nu}\,\frac{a^+ P_n^{\nu-\alpha,\,\alpha-1}(v)}{(y - a)^{1-\alpha}(b - y)^{\alpha-\nu}} + \frac{a_-}{\nu}\int_a^b \frac{\operatorname{sign}(x - y) P_n^{\nu-\alpha,\,\alpha-1}(v)\,dy}{(y - a)^{1-\alpha}(b - y)^{\alpha-\nu}|x - y|^{\nu}} =$$

$$= \frac{\pi\Gamma(\nu + n) P_n^{\alpha-1,\,\nu-\alpha}(u)}{\Gamma(1 + \nu)\sin\pi\alpha\sin\pi(\alpha - \nu)n!} - \delta_{n0}\frac{a^+\Gamma(\alpha)\Gamma(1 + \nu - \alpha)}{(b - a)^{-\nu}\nu\Gamma(\nu + 1)}$$

(δ_{nm} is the Kronecker symbol)

Passing now to the limit $\nu \to 0$ separately in the cases $n = 0$ and $n \neq 0$ (an analogous operation but for another purpose was performed in [4]) we arrive at a spectral relation of the Π'-kernel of the form

$$\Pi(x, y) = \frac{1}{2}\operatorname{sign}(x - y) + \frac{\operatorname{tg}\pi\alpha}{\pi}\ln\frac{1}{|x - y|} \tag{3.26}$$

In the case $a = -1$, $b = 1$, this was obtained and applied to a contact problem in [7]. In an analogous way, from (3.20) one may obtain also the bilinear expansion for the Π-kernel (3.26).

4. The construction of Π-kernels for a semi-infinite interval.

In order to construct Π-kernels on a semi-infinite interval by this approach, one must take a representation for the kernel analogous to (3.1). Another family of Π-kernels may be obtained by taking as basis the following representation:

$$K(x, y) = \int_{\max(x, y)}^{\infty} K_{+}(s - x) K_{-}(s - y) \rho(s)\, ds \tag{4.1}$$

However the same results may be obtained more quickly by the following formal procedure. In the spectral relation (3.15) we set $a = 0$, $\sigma = -b$, after which we let b tend to ∞. As a result of using the limiting equations

$$\lim_{b\to\infty} P_n^{\alpha,\, b+\beta}(1 - 2x / b) = L_n^{(\alpha)}(x)$$

$$\lim_{b\to\infty} (b - x)^b = e^{-x}, \qquad \lim_{|z|\to\infty} \frac{\Gamma(z + a)}{\Gamma(z) z^a} = 1 \tag{4.2}$$

we shall have

$$\int_0^{\infty} \frac{\Pi_{\infty}(x, y)}{e^y} L_n^{(1+\rho-\beta)}(y)\, dy = \frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(1+\rho+n)}{\Gamma(2+\rho-\alpha+n) x^{\alpha-\rho-1}} L_n^{(1+\rho-\alpha)}(x) \tag{4.3}$$

Here $L_n^{(\alpha)}(x)$ is a Chebyshev-Laguerre polynomial and the Π-kernel has the form

$$\Pi_{\infty}(x, y) = \int_0^{\chi} \frac{e^s s^{\rho}\, ds}{(x - s)^{\alpha}(y - s)^{\beta}} \qquad (\operatorname{Re}(1+\rho,\ 1-\alpha,\ 1-\beta) > 0,\ \chi = \min(x, y)) \tag{4.4}$$

The constructed Π-kernel may be expressed in terms of a degenerate hypergeometric function $\Phi_1(\alpha, \beta, \gamma; x, y)$ of two variables [21]. For this, in the case $x < y$, it suffices to perform the substitution $s = xt$ in (4.4), expand the exponent in a series, and finally expand the remaining integral in a series in x / y. As a result we have

$$\Pi_{\infty}(x, y) = \frac{\Gamma(1+\alpha)\Gamma(1+\rho)}{\Gamma(2+\alpha+\rho)} \Phi_1(1+\rho, \beta, \alpha+\rho+2, x, x/y) \qquad (x<y)$$

For $x > y$ one should replace α by β and x by y on the right.

One could also arrive at these results by starting directly from the integral representation (3.1) for $a = 0$. On the other hand we arrive at the results obtainable on the basis of representation (4.1) if in this same spectral relation (3.15) and formula (3.11) with $a = 0$, $\rho = b$, we make the change of variables $s = b - t$, $x = b - \xi$, $y = b - \eta$, after which we let $b \to \infty$. As a result of the use of (4.2) we shall have

$$\int_0^{\infty} \frac{\Pi^{\infty}(\xi, \eta)}{\eta^{1+\sigma-\beta}} L_n^{(\beta-\sigma-1)}(\eta)\, d\eta = \frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\beta-\sigma+n)}{\Gamma(1-\sigma+n) e^{\xi}} L_n^{(\alpha-\sigma-1)}(\xi) \tag{4.5}$$

And in this,

$$\Pi^{\infty}(\xi, \eta) = \int_{\max(\xi,\eta)}^{\infty} \frac{e^{-s} s^{\sigma}\, ds}{(s-\xi)^{\alpha}(s-\eta)^{\beta}}, \qquad \operatorname{Re}(1-\alpha,\ 1-\beta,\ \alpha-\sigma,\ \beta-\sigma) > 0 \tag{4.6}$$

Not dwelling on the many particular cases of Π-kernels (4.4) and (4.6), we merely note that the latter for $\sigma = 0$ is expressible in terms of a degenerate hypergeometric function [21] of second kind. Actually, considering, for example, the case $\xi > \eta$, after obvious changes of variable and use of formula 9.211 (4) from [20], one may see that

$$\Pi^{\infty}(x, y) = e^{-x}(x - y)^{1-\alpha-\beta}\,\Gamma(1-\alpha)\,\Psi(1-\alpha,\ 2-\alpha-\beta;\ x - y) \qquad (x > y)$$

for $y > x$ on the right side of this latter formula one should replace y by x and α by β. If $\alpha = \beta$, the both cases ($x > y$ and $x < y$) are governed by the same formula, which permits the spectral relation (4.5) to be written in the form

$$\int_0^\infty \frac{\Psi(\mu, 2\mu, |\xi-\eta|)\, L_n^{(-\mu)}(\eta)\, d\eta}{\eta^\mu \sqrt{e^\eta e^{|\xi-\eta|}}\, |\xi-\eta|^{1-2\mu}} = \frac{\Gamma(\mu)\,\Gamma(1-\mu+n)}{n!\, e^{\xi/2}} L_n^{(-\mu)}(\xi) \qquad (4.7)$$

$$(\mu = 1-\alpha,\ 0 < \operatorname{Re}\mu < 1)$$

If we here set $\mu = \nu + 1/2$, make the changes $\xi = 2x$, $\eta = 2y$ and use the known ([21], p. 253) formula for the MacDonald function $K_\nu(z)$, then we obtain a spectral relation for the П-kernel of the form

$$K_\nu(|x-y|)\,|x-y|^{-\nu} \qquad (4.8)$$

which was first found and applied to certain problems in [4, 6].

It should be noted that the bilinear expansion for the П-kernel (4. 8) given in [4] is only formal, since its Hilbert-like nature was not proved. Moreover it may be shown that the П-kernel (4. 8) for $\nu = 0$ will not be Hilbert-like and the bilinear expansion indicated in [4] converges in a weaker sense than the L_2 convergence.

BIBLIOGRAPHY

1. Klubin, P. I., Analysis of beam-like and circular plates on elastic foundation. Inzh. sb., Vol. 12, 1952.
2. Popov, G. Ia., On an approximate method for solving certain plane contact problems of the theory of elasticity. Izv. Akad. Nauk ArmSSR, ser. fiz.-matem. n., Vol. 14, №3, 1961.
3. Popov, G. Ia., The contact problem of the theory of elasticity for the case of a circular area of contact. PMM Vol. 26, №1, 1962.
4. Popov, G. Ia., Some properties of classical polynomials and their application to contact problems. PMM Vol. 27, №5, 1963, PMM Vol. 28, №3, 1964.
5. Aleksandrov, V. M., On the approximate solution of a class of integral equations. Izv. Akad. Nauk ArmSSR, ser. fiz.-matem. n., Vol. 17, №2, 1964.
6. Popov, G. Is., On an approximate method of solving the integral equations of the diffraction of electromagnetic waves on a strip of finite width. Zh. tekhn. fiz., Vol. 35, №3, 1965.
7. Popov, G. Ia., Plane contact problem of the theory of elasticity with bonding or frictional forces. PMM Vol. 30, №3, 1966.
8. Lutchenko, S. A., On the impression of a stamp into the lateral surface of an elastic foundation in the form of a wedge. Prikl. mekhan., Vol. 2, №12, 1966.
9. Popov, G. Ia., Impression of a punch on a linearly deforming foundation taking into account friction forces. PMM Vol. 31, №2, 1967.

15. Popov, G. Ia., Certain new relations for Jacobi polynomials. Sib. matem. zh., Vol. 8, №6, 1967.
16. Shtaerman, I. Ia., Contact Problems in the Theory of Elasticity. M.-L., Gostekhteorizdat, 1949.
17. Tricomi, F. G., Integral Equations (Russ. transl.), M., Izd. inostr. lit., 1960.
18. Szego, G., Orthogonal Polynomials (Russian translation). M., Fizmatgiz, 1962.
19. Tsel'nik, D. S., Symmetric contact forms of a jet and the surface of a heavy fluid. Izv. Akad. Nauk SSSR, MZhG, №3, 1966.
20. Gradshtein, I. S. and Ryzhik, I. M., Tables of Integrals, Sums, Series, and Products. M., Fizmatgiz, 1962.
21. Bateman, G. and Erdelyi, A., Higher Transcendental Functions. Hypergeometric Functions. Legendre Functions (Russian translation). M., "Nauka", 1965.
22. Bateman, G. and Erdelyi, A., Higher Transcendental Functions. Bessel Functions, Parabolic Cylinder Functions, Orthogonal Polynomials (Russian translation). M. "Nauka", 1966.

Translated by P. F.

CONCENTRATED FORCE IN A TRANSVERSALLY-ISOTROPIC HALF-SPACE AND IN A COMPOSITE SPACE

PMM Vol. 33, №3, 1969, pp. 532-537
V. A. SVEKLO
(Kaliningrad)
(Received October 22, 1968)

The problem of the effect of a concentrated force in an isotropic (orthotropic) space has been examined in [1–3].

The problem is investigated below by the method of complex Smirnov-Sobolev solutions, generalized to a system of differential equations.

The results obtained are of elementary nature just for a transversally isotropic solid.

1. Complex solutions of the equilibrium equations. If the potentials φ, ψ, χ are introduced by assuming

$$u = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x}, \quad w = \frac{\partial\chi}{\partial z} \tag{1.1}$$

then the equilibrium equations of a transversally isotropic body under the condition that the z-axis is along the axis of elastic symmetry become

$$\frac{\partial L_1}{\partial x} + \frac{\partial Q}{\partial y} = 0, \quad \frac{\partial L_1}{\partial y} - \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial L_2}{\partial z} = 0 \tag{1.2}$$

$$L_1 = A\,\Delta\varphi + Ld^2\varphi\,/\,dz^2 + (L+F)d^2\chi\,/\,dz^2$$

$$L_2 = (L+F)\,\Delta\varphi + L\,\Delta\chi + Cd^2\chi\,/\,dz^2 \tag{1.3}$$

$$Q = N\,\Delta\psi + Ld^2\psi\,/\,dz^2, \qquad \Delta = d^2/dx^2 + d^2/dy^2$$

Here A, L, F, N, C are elastic constants [4]. Let us construct the solution of the system (1.2) in the form

$$\varphi = \operatorname{Re}\varphi^\circ(\theta), \qquad \psi = \operatorname{Re}\psi^\circ(\theta), \qquad \chi = \operatorname{Re}\chi^\circ(\theta) \tag{1.4}$$

The variable θ is defined by the relationship